Chapter
Simplex Method

After reading this chapter, you should be able to:

1. *Formulate constrained optimization problems as a linear program*
2. *Solve linear programs with graphical solution approaches*
3. *Solve constrained optimization problems using simplex method*

What is linear programming?

Linear programming is an optimization approach that deals with problems that have specific constraints. The one-dimensional and multi-dimensional optimization problems previously discussed did not consider any constraints on the values of the independent variables. In linear programming, the independent variables which are frequently used to model concepts such as availability of resources or required ratio of resources are constrained to be more than, less than or equal to a specific value.

The simplest linear program requires an objective function and a set of constraints. The objective function is either a maximization or a minimization of a linear combination of the independent variables of the problem and is expressed as (for a maximization problem)

$$\max z = c_1 x_1 + c_2 x_2 + ... + c_n x_n$$

where c_i expresses the contribution (e.g. cost, profit etc) of each unit of x_i to the objective of the problem, and x_i are the independent or more commonly referred to as the decision variables whose values are determined by the solution of the problem.

The constraints are also a linear combination of the decision variables commonly expressed as an inequality of the form

$$a_{i1} x_1 + a_{i2} x_2 + ... + a_{in} x_n \le b_i$$

where a_{ij} and b_i are constant coefficients determined from the problem description as they relate to the constraints on the availability, interaction, and use of the resources.

Example 1

A woodworker builds and sells band-saw boxes. He manufactures two types of boxes using a combination of three types of wood, maple, walnut and cherry. To construct the Type I box, the carpenter requires 2 board foot (bf) (The board foot is a specialized unit of measure for the volume of lumber. It is the volume of a one-foot length of a board one foot wide and one inch thick) maple and 1 bf walnut. To construct the Type II box, he requires 3 bf of cherry

and 1 bf of walnut. Given that he has 10 bf of maple, 5 bf of walnut and 11 bf of cherry and he can sell Type I of box for $120 and Type II box for $160, how many of each box type should he make to maximize his revenue? Assume that the woodworker can build the boxes in any size, therefore fractional solutions are acceptable.

Solution

The decision variables in this problem are the number of Type I and II boxes to be built. They are denoted by x_1 and x_2 respectively. Since the goal is to maximize revenues and the revenues are a function of the number of boxes of each type sold, we can represent the objective function as

$$\max z = 120x_1 + 160x_2$$

One of the constraints in this problem is availability of different types of wood. Therefore, based on the number of boxes produced, the sum of the total wood requirement must be less than or equal to the available amount of wood for each type. We can represent this type of constraint with three inequalities referring to maple, cherry and walnut respectively as follows:

$$2x_1 \leq 10$$

$$3x_2 \leq 11$$

$$x_1 + x_2 \leq 5$$

In addition, there are the non-negativity constraints which ensure that our solution does not have negative number of boxes. These constraints are shown as

$$x_1, x_2 \geq 0$$

**Graphical Solutions to Linear Programs**

Linear programs of two or three dimensions can be solved using graphical solutions. While graphical solutions are not useful in addressing realistic size problems, they are particularly helpful in providing an intuitive explanation to the algebraic methodologies used to solve larger linear programs using computer algorithms. The graphical solution to linear programs is best explained by using an example.

Example 2

Provide a graphical solution to the linear program in Example 1.

Solution

For a linear inequality of the form $f(x_1, x_2) \leq b$ or $f(x_1, x_2) \geq b$, the points that satisfy the inequality includes the points on the line and the points on one side of the line. For example for the inequality $2x_1 \leq 10$, the shaded region in Figure 1 shows the points that satisfy this inequality. To determine which side of the line satisfies the inequality, simply test a single point in each region, such as the origin (0, 0) which satisfies the constraint and lies on the right side of the line in the shaded region.
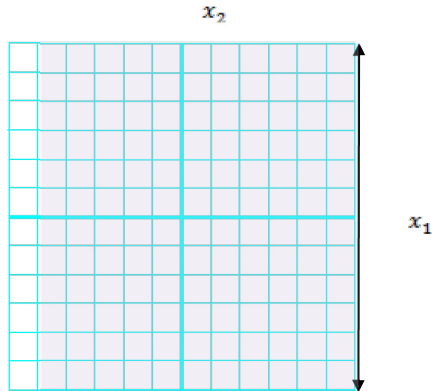
Figure 1. Graphical representation of the points satisfying $2x_1 \leq 10$.

The set of points that satisfy all the constraints, including non-negativity constraints, from Example 1 are shown in Figure 2. The region which contains the points that satisfies all the constraint in a linear program is referred to as the feasible region.
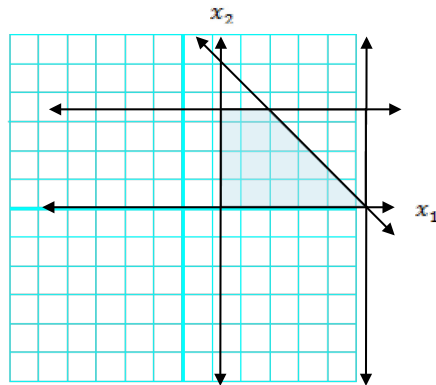
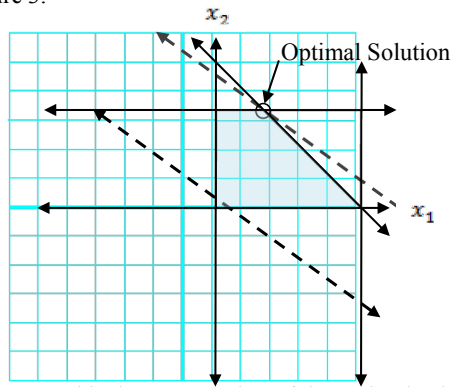**Figure 2.** Graphical representation of the feasible region.

The objective function can also be represented by a line referred to as the isoprofit line (isocost line for minimization problems). To determine this line, simply assume a value for z such as $z = 0$. Then the objective function can be written as

$$0 = 120x_1 + 160x_2$$

$$x_2 = -\frac{120}{160}x_1 = -\frac{3}{4}x_1$$

where the isoprofit line has a slope of $-\frac{3}{4}$. The isoprofit line is shown as a dashed line through the origin in Figure 3. To determine the optimal solution, the isoprofit line is moved parallel to the original line drawn with slope $-\frac{3}{4}$ in the direction that increases z until the

last point intersecting the feasible region is obtained. Such a point is reached at a single point $\left(\dfrac{4}{3}, \dfrac{11}{3}\right)$ as shown in Figure 3.

Figure 3. Graphical representation of the optimal solution.

At the optimal solution, the value of the objective function is calculated as

$$120 \times \frac{4}{3} + 160 \times \frac{11}{3} = 746\frac{2}{3}$$

The optimal solution when substituted back into the inequalities representing the structure of the problem reveals some additional important information about the problem. Below is the original set of constraints where the optimal solution to the problem is substituted in place of the decision variables. Note that the last two equations are now equalities indicating that the availability of the resources associated with these constraints (cherry and walnut) are preventing us from improving the value of the objective function. Such constraints are referred to as binding constraints. Note also that in the graphical solution, the optimal solution lies at the intersection of the binding constraints. On the other hand, the first inequality is a nonbinding constraint in the sense that the left-hand and the right-hand side of the constraint are unequal and this constraint does not pose a limitation to the optimal solution. In other words, if want to increase our revenues, we need to look into increasing the availability of cherry and walnut and not maple.

$$2 \times \frac{4}{3} < 10$$

$$3 \times \frac{11}{3} = 11$$

$$\frac{4}{3} + \frac{11}{3} = 5$$

Solutions to Linear Programs

Solutions to linear programs can be one of two types as follows:

1. Unique solution:

As seen in the solution to Example 2, there is a single point in the feasible region for which the maximum (or minimum in a minimization problem) value of the objective function is attainable. In graphical solutions, these points lie at the intersection of two or more lines which represent the constraints.

2. Alternate Solutions:

If the isoprofit (isocost) line is parallel to one of the lines representing the constraints, then the intersection would be an infinite number of points. In this case, any of such points would produce the maximum (minimum) value of the objective function.

A set of points S is said to be a convex set if the line segment joining any pair of points in S is also completely contained in S. For example, the feasible region shown in Figure 2 is a convex set. This is no coincidence. It can be shown that the feasible region of any linear program is a convex set.

Figure 4 shows the feasible region of Example 2 and highlights the corner points (also known as extreme points) of the convex set which occur where two or more constraints intersect within the feasible region. These extreme points are of special importance. Any linear program that has an optimal solution has an extreme point that is optimal. This is a very important result because it greatly reduces the number of points which may be optimal solutions to the linear program. For example, the entire feasible region shown in Figure 2 contains an infinite number of points, however the feasible region contains only four extreme points which may be the optimal solution to the linear program.
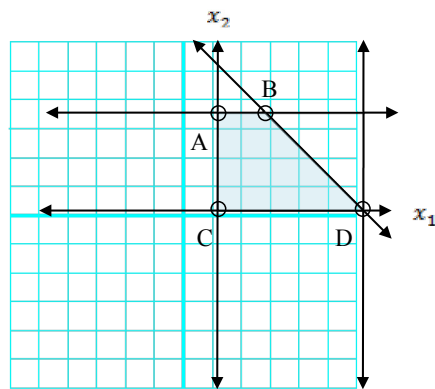
Figure 4. Graphical representation of the feasible region and its extreme points.

Once all the extreme points are determined, finding the optimal solution is trivial in the sense that the value of the objective function at each of these points can be calculated and, depending on the goal of the objective function, the extreme point resulting in the minimum or the maximum value is selected as the optimal solution. The simplex method which is the topic of next section is a much more efficient way of evaluating the extreme points in a convex set to determine the optimal solution.

The Simplex Method

<u>Converting a linear program to Standard Form</u>
Before the simplex algorithm can be applied, the linear program must be converted into standard form where all the constraints are written as equations (no inequalities) and all variables are nonnegative (no unrestricted variables). This process of converting a linear program to its standard form requires the addition of slack variable s_i which represents the amount of the resource not used in the ith $\leq$ constraint. Similarly, $\geq$ constraints can be converted into standard form by subtracting excess variable e_i.

The standard form of any linear program can then be represented by the following linear system with n variables (including decision, slack and excess variables) and m constraints.

$$
\begin{array}{ll}
\max z = & \\
(or \min) & c_1x_1 + \quad c_2x_2 + \quad ...+ \quad c_nx_n \\
s.t. & a_{11}x_1 + \quad a_{12}x_2 + \quad ...+ \quad a_{1n}x_n \quad = b_1 \\
& a_{21}x_1 + \quad a_{22}x_2 + \quad ...+ \quad a_{2n}x_n \quad = b_2 \\
& \quad ... \qquad ... \qquad ...+ \qquad ... \qquad ... \\
& a_{m1}x_1 + \quad a_{m2}x_2 + \quad ...+ \quad a_{mn}x_n \quad = b_m
\end{array}
$$

$$x_i \geq 0 \quad (i = 1,2,...,n)$$

Example 3

Convert the linear program in Example 1 to its standard form.
Solution

For convenience, the linear program is reproduced below.

$$Max \ Z = 120x_1 + 160x_2$$
$$2x_1 \leq 10$$
$$3x_2 \leq 11$$
$$x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0$$

To convert the first constraint form an inequality to equality, we introduce the first slack variable s_1 where

$$s_1 = 10 - 2x_1 \text{ or } 2x_1 + s_1 = 10.$$

Similarly after introducing s_2 and s_3, we can convert the linear program into standard form as follows:

$$\begin{aligned} \max z = \quad & 120x_1 \quad +160x_2 \\ \text{s.t.} \quad & 2x_1 \qquad\qquad + s_1 \qquad\qquad\quad = 10 \\ & \qquad +3x_2 \qquad\quad + s_2 \qquad\quad = 11 \\ & x_1 \quad + x_2 \qquad\qquad + s_3 \quad = 5 \\ & x_1, x_2, s_1, s_2, s_3 \quad \geq 0 \end{aligned}$$

Basic and Nonbasic Variables, and Basic Feasible Solutions

If we define

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix},$$

the constraints of the standard form of a linear program can be simply represented by a system of simultaneous equations $A\mathbf{x} = \mathbf{b}$.

A basic solution to system of m linear equations with n unknowns is found by setting $n-m$ variables to zero and solving the m equations for the remaining m variables. The variables with zero values are referred to as the nonbasic variables and the remaining m variables are called the basic variables. Note that the choice of different nonbasic variables will lead to different solutions. If all basic variables are nonnegative, the solution is called a basic feasible solution. The optimum solution will be one of the basic feasible solutions. Let us illustrate this with an example.

Example 3

Determine a basic feasible solution for the linear program in Example 1.
Solution

The system of equation representing the constraints for this linear program is as follows:

$$\begin{aligned} 2x_1 \qquad\qquad + s_1 \qquad\qquad\quad = 10 \\ +3x_2 \qquad\quad + s_2 \qquad\quad = 11 \\ x_1 \quad + x_2 \qquad\qquad + s_3 \quad = 5 \end{aligned}$$

where $n = 5$ and $m = 3$. To obtain a basic feasible solution we need to set $n - m = 2$ nonbasic variables to zero and solve the remaining system of 3×3 linear equations. Let us start with setting values of x_1 and x_2 to zero. We can easily see that the solution to the system becomes

$s_1 = 10$

$s_2 = 11$

$s_3 = 5$

In this solution, all basic variables are nonnegative; therefore the solution is a basic feasible solution.

Relationship between extreme points of a feasible region and basic feasible solutions

To establish the relationship between basic feasible solutions and extreme points of the feasible region, refer to Figure 4. The above basic feasible solution corresponds to the extreme point C at the origin since in this basic feasible solution $x_1 = 0$ and $x_2 = 0$. Alternatively, if we set the values of s_3 and x_2 to zero, we see that we obtain the basic feasible solution where $x_1 = 5, s_1 = 5,$ and $s_2 = 11$ which corresponds to the extreme point D in Figure 4.

There is a special relationship between extreme points C and D arising from their adjacency that is relevant to the simplex method. For a linear program with m constraints, two basic feasible solutions are adjacent if they have $m - 1$ basic variables in common. In the basic feasible solutions corresponding to adjacent points C and D, the $m - 1$ common basic variables are $s_1 = 5,$ and $s_2 = 11$.

The Simplex Algorithm

The simplex algorithm, instead of evaluating all basic feasible solutions (which can be prohibitive even for moderate-size problems), starts with a basic feasible solution and moves through other basic feasible solutions that successively improve the value of the objective function. The algorithm terminates once the optimal value is reached. Below we present a step-wise description of the simplex algorithm.

1. Convert the linear program into standard form.
2. Obtain a basic feasible solution from the standard form.
3. Determine if the basic feasible solution is optimal.
4. If the current basic feasible solution is not optimal, select a nonbasic variable that should become a basic variable and basic variable which should become a nonbasic variable to determine a new basic feasible solution with an improved objective function value.
5. Use elementary row operations to solve for the new basic feasible solution. Return to Step 3

Steps 1 and 2 of the algorithm have been previously discussed. Steps 3, 4 and 5 of the algorithm are best executed with the help of a tableau which is simply a table with a particular format that shows a summary of the key information regarding the linear program. For example the tableau shown in Table 1 below corresponds to the linear program described in Example 1 and the basic feasible solution in Example 3. There are several things to note about Table 1.

1. The first row of the table (also called row 0) corresponds to the objective function where all the variables are on the left-hand side following the format
2. $z - 120x_1 - 160x_2 = 0$
3. The basic feasible solution corresponds to the solution in Example 3. In addition note that variable $z = 0$ is also considered as a basic variable.

4. In this particular example, the initial tableau where the decision variables x_1 and x_2 are considered as nonbasic variables leads to a basic feasible solution due to the fact that all the right hand side variables are nonnegative.
5. The tableau is in proper form which means the solution can be read directly by looking at the tableau and the RHS values. For a tableau to be in proper form it must meet all the following requirements:

one basic variable per row

the coefficient of all basic variables are +1 and the coefficients above and below the basic variables are zero

z is the basic variable for row 0

Table 1. The initial tableau for example on in proper form

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	-160	0	0	0	0	
s_1	0	2	0	1	0	0	10	None
s_2	0	0	3	0	1	0	11	$11/3$
s_3	0	1	1	0	0	1	5	5

In step 3, to determine if a basic feasible solution is optimal, we need to determine if any of the nonbasic variables (who has value zero) can be increased to improve the value of the objective function. For example, in Table 1, since $z = 120x_1 + 160x_2$, increasing either one of the nonbasic variables x_1 and x_2 would increase the value of the objective function value. In the tableau, this equates to looking for negative coefficients in row 0 due to the format the objective function is written. The basic feasible solution shown in Table 1 is therefore not optimal since the coefficients of x_1 and x_2 are less than zero.

To improve the solution, we can increase the value of either x_1 or x_2. We choose to increase x_2 since the value of the objective function increases at a higher rate (160 vs. 120 per unit of increase). The nonbasic variable with the most negative coefficient (in a maximization problem) in row 0, in this case x_2, is called the entering variable and is always selected as the nonbasic variable that becomes a basic variable.

The basic variable that is replaced by the entering variable, also called the leaving variable, is determined by looking at the values in the "Ratio" column in the tableau. The values in this column are simply the ratio of the RHS values divided by the coefficient of the entering variable in that row. The leaving variable is selected to be the basic variable in the row with the smallest ratio. This is the highest value that the entering variable can have and still result in a basic feasible solution. For the tableau shown in Table 1, the leaving variable is s_2 in row 3.

Once the entering and leaving variables are determined, we use elementary row operations (add link?) (EROs) to make the entering variable a basic variable in the row of the leaving variable by making its coefficient 1 in that row and 0 in all other rows. For example, for Table 1 where the entering and leaving variables are x_2 and s_2 respectively, after the

EROs, the tableau is shown in Table 2. The tableau shows a new basic feasible solution (note that all RHS are nonnegative) where

$$s_1 = 10$$
$$x_2 = 11\!\!\not\!/_3$$
$$s_3 = 4\!\!\not\!/_3$$

This basic feasible solution corresponds to the adjacent extreme point A in Figure 4 with coordinates $x_1 = 0$ and $x_2 = 11\!\!\not\!/_3$ and objective function value $1760\!\!\not\!/_3$.

Table 2. The tableau for the basic feasible solution corresponding to extreme point A in proper form.

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	0	0	$160\!\!\not\!/_3$	0	$1760\!\!\not\!/_3$	
s_1	0	2	0	1	0	0	10	5
x_2	0	0	1	0	$\tfrac{1}{3}$	0	$11\!\!\not\!/_3$	None
s_3	0	1	0	0	$-\tfrac{1}{3}$	1	$4\!\!\not\!/_3$	$4\!\!\not\!/_3$

After a new basic feasible solution is obtained, the algorithm returns to Step 3 to check if the new basic feasible solution is optimal. This cycle continues until the objective function value cannot be increased by increasing the value of any of the nonbasic variables. In other words, in a maximization problem, this is the same as having no negative valued coefficients in row 0.

A note about minimization problems:
 It is important to note that the optimality condition of no negative valued coefficients in row 0 is only applicable in maximization problems. In a minimization problem, the optimality condition exists when none of the coefficients in row 0 are positive. Furthermore, in minimization problems, the entering variable is chosen to be the nonbasic variable with the highest positive coefficient in row 0.
Let us illustrate the simplex algorithm by solving the problem presented in Example 1.

Example 4

Solve the linear program in Example 1using the simplex algorithm.
Solution

Step 1:
Convert the linear program into standard form.
The linear program in standard form is

$$\max z = 120x_1 + 160x_2$$
$$s.t. \quad 2x_1 \qquad\qquad + s_1 \qquad\qquad\qquad = 10$$
$$+3x_2 \qquad\qquad + s_2 \qquad\quad = 11$$
$$x_1 \quad + x_2 \qquad\qquad\qquad + s_3 \quad = 5$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Step 2:
Obtain a basic feasible solution from the standard form.
Previously we have shown that the solution where $x_1 = 0$ and $x_2 = 0$ is a basic feasible solution so we will start the algorithm here.

Step 3:
Determine if the basic feasible solution is optimal.
At this step we create the tableau for this basic feasible solution which was initially shown in Table 1. For convenience the table is reproduced as Table 3.

Table 3. The initial tableau in proper form

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	-160	0	0	0	0	
s_1	0	2	0	1	0	0	10	None
s_2	0	0	3	0	1	0	11	$11/3$
s_3	0	1	1	0	0	1	5	5

Step 4:
If the current basic feasible solution is not optimal, select a nonbasic variable that should become a basic variable and basic variable which should become a nonbasic variable to determine a new basic feasible solution with an improved objective function value.
The current solution is not optimal. There are negative coefficients in row 0. Since x_2 has the most negative coefficient in row 0 and s_2 has the lowest ratio, the entering and the leaving variables are x_2 and s_2, respectively.

Step 5:
Use elementary row operations to solve for the new basic feasible solution. Return to Step 3
The new basic feasible solution is shown in Table 4, which is the same as Table 2.

Table 4. The tableau for the new basic feasible solution in the first iteration

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	-120	0	0	$160/3$	0	$1760/3$	
s_1	0	2	0	1	0	0	10	5
x_2	0	0	1	0	$1/3$	0	$11/3$	None
s_3	0	1	0	0	$-1/3$	1	$4/3$	$4/3$

Step 3:
Determine if the basic feasible solution is optimal.
The basic solution in Table 4 is still not optimal as the objective function value can be increased by increasing the value of x_1.

Step 4:
If the current basic feasible solution is not optimal, select a nonbasic variable that should become a basic variable and basic variable which should become a nonbasic variable to determine a new basic feasible solution with an improved objective function value.
In the second iteration, since x_1 has the most (and only) negative coefficient in row 0 and s_3 has the lowest ratio, the entering and leaving variables are x_1 and s_3, respectively.

Step 5:
Use elementary row operations to solve for the new basic feasible solution. Return to Step 3
The new basic feasible solution is shown in Table 5.

Table 5. The tableau for the basic feasible solution in the second iteration .

Basic	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratio
Z	1	0	0	0	$40/3$	120	$2240/3$	
s_1	0	0	0	1	$2/3$	-2	$22/3$	
x_2	0	0	1	0	$1/3$	0	$11/3$	
x_1	0	1	0	0	$-1/3$	1	$4/3$	

Step 3:
Determine if the basic feasible solution is optimal.
Since there are no negative coefficients in row 0, we have reached the optimal solution where the objective function value is $2240/3$ and

$$s_1 = 22/3$$
$$x_2 = 11/3$$
$$x_1 = 4/3$$
$$s_2 = s_3 = 0$$

Note that all these values can be read from the tableau shown in Table 5. This solution also corresponds to the extreme point B in Figure 4 which was also determined to be optimal using the graphical solution approach.

Finally, the woodworker should build $4/3$ Type I boxes and $11/3$ Type II boxes to maximize his revenue to $746.67.

Chapter Bibliography

Topic	Simplex Method
Summary	Textbook notes for the Simplex method
Authors	Ali Yalcin, Autar Kaw,

QUESTION

(a) Use the simplex method to find *all* optimal solutions of the following linear programming problem.

Maximize $z = -2x_1 + 7x_2 + 4x_3$
subject to $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$
$$2x_1 - 5x_2 \leq 11$$
$$-x_1 + 3x_2 + x_3 = 7$$
$$x_1 - 8x_2 + 4x_3 \geq 33.$$

(b) A company needs to lease warehouse space over the next five months. The requirements for space in thousands of square feet are shown in the following table.

	Month 1	Month 2	Month 3	Month 4	Month 5
Required space	45	30	50	20	60

It is possible to lease the exact space needed on a month-by-month basis. However, the cost per month becomes less as the period of lease increases, so it may be more economical to lease the maximum space needed for the entire five-month period. A variety of intermediate strategies are also possible which may involve starting two or more leases, each having a different leasing period, in some months. The cost per month of different length leases is shown in the following table.

Leasing period (months)	1	2	3	4	5
Cost (£) per 1000 square feet per month	600	480	420	380	350

Write down a linear programming formulation (but do not attempt to solve it) for the problem of finding a leasing policy so that the space requirements for the next five months are met at minimum total leasing cost.

ANSWER

(a) Add slack variables s_1, s_2 and artificial variables a_1, a_2.

Basic	z'	z	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	0	0	2	-5	0	1	0	0	0	11
a_1	0	0	-1	3	1	0	0	1	1	7
a_2	0	0	1	-8	4	0	-1	0	1	33
	1							$+1$	$+1$	0
	1	0	0	5	-5	0	1	0	0	-40
	0	1	2	-7	-4	0	0	0	0	0

1

Basic	z'	z	x_1	x_2	x_3	s_1	s_2	a_1	a_2	
s_1	0	0	2	-5	0	1	0	0	0	11
x_3	0	0	-1	3	1	0	0	1	1	7
a_2	0	0	5	-20	0	0	-1	-4	1	5
	1	0	-5	20	0	0	1	5	0	-5
	0	1	-2	5	0	0	0	4	0	28

Basic	z'	$-z$	$-x_1$	$-x_2$	x_3	s_1	s_2	a_1	a_2	
s_1	0	0	0	3	0	1	$\frac{2}{5}$	$\frac{8}{5}$	$-\frac{2}{5}$	9
x_3	0	0	0	-1	1	0	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1
	-1	0	0	0	0	0	0	1	1	0
	0	1	0	-3	0	0	$-\frac{2}{5}$	$\frac{12}{5}$	$\frac{2}{5}$	30

Basic	x	x_1	x_2	x_3	s_1	s_2	
x_2	0	0	1	0	$\frac{1}{3}$	$\frac{2}{15}$	3
x_3	0	0	0	1	$\frac{1}{3}$	$-\frac{1}{15}$	11
x_1	0	1	0	0	$\frac{4}{3}$	$\frac{1}{3}$	13
	1	0	0	0	1	0	39

Solution is $x_1 = 13$, $x_2 = 3$, $x_3 = 11$, $z = 39$

Perform another iteration to find an alternative optimal solution.

Basic	x	x_1	x_2	x_3	s_1	s_2	
s_2	0	0	$\frac{15}{2}$	0	$\frac{5}{2}$	1	$\frac{45}{2}$
x_3	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{11}{2}$
	1	0	0	0	1	0	39

An alternative optimal solution is $x_1 = \frac{11}{2}$, $x_2 = 0$, $x_3 = \frac{25}{2}$, $z = 39$.

Thus $(x_1, x_2, x_3) = \alpha(13, 3, 11) + (1 - \alpha)\left(\frac{11}{2}, 0, \frac{25}{2}\right)$ for $0 \le \alpha \le 1$ is the class of all optimal solutions.

(b) Let x_{11}, x_{12}, x_{13}, x_{14}, x_{15} be the number of square feet leased at the start of month 1 for 1, 2, 3, 4, 5 months respectively.

Variables

x_{21}, x_{22}, x_{23}, x_{24}	for month 2
x_{31}, x_{32}, x_{33}	for month 3
x_{41}, x_{42}	for month 4
x_{51}	for month 5

are defined similarly.

Maximize

$$z = \quad 600(x_{11} + x_{21} + x_{31} + x_{41} + x_{51})$$

$$+960(x_{12} + x_{22} + x_{32} + x_{42})$$
$$+1260(x_{13} + x_{23} + x_{33})$$
$$+1520(x_{14} + x_{24}) + 1750x_{15}$$

subject to $x_{ij} \geq 0$ all i, j.

$$
\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} + x_{15} &\geq 45 \\
x_{12} + x_{13} + x_{14} + x_{15} & \\
+x_{21} + x_{22} + x_{23} + x_{24} &\geq 30 \\
x_{13} + x_{14} + x_{15} & \\
+x_{22} + x_{23} + x_{24} & \\
+x_{31} + x_{32} + x_{33} &\geq 50 \\
x_{14} + x_{15} + x_{23} + x_{24} & \\
+x_{32} + x_{33} + x_{41} + x_{42} &\geq 20 \\
x_{15} + x_{24} + x_{33} + x_{42} + x_{51} &\geq 60
\end{aligned}
$$