Integration – Basics and its Application

ASHA RANI

B.COM(HONS.) IVth SEMESTER

BUSINESS MATHEMATICS (PAPER- BCH 4.2)

SECTION H

Session Details:

This session covers the following :

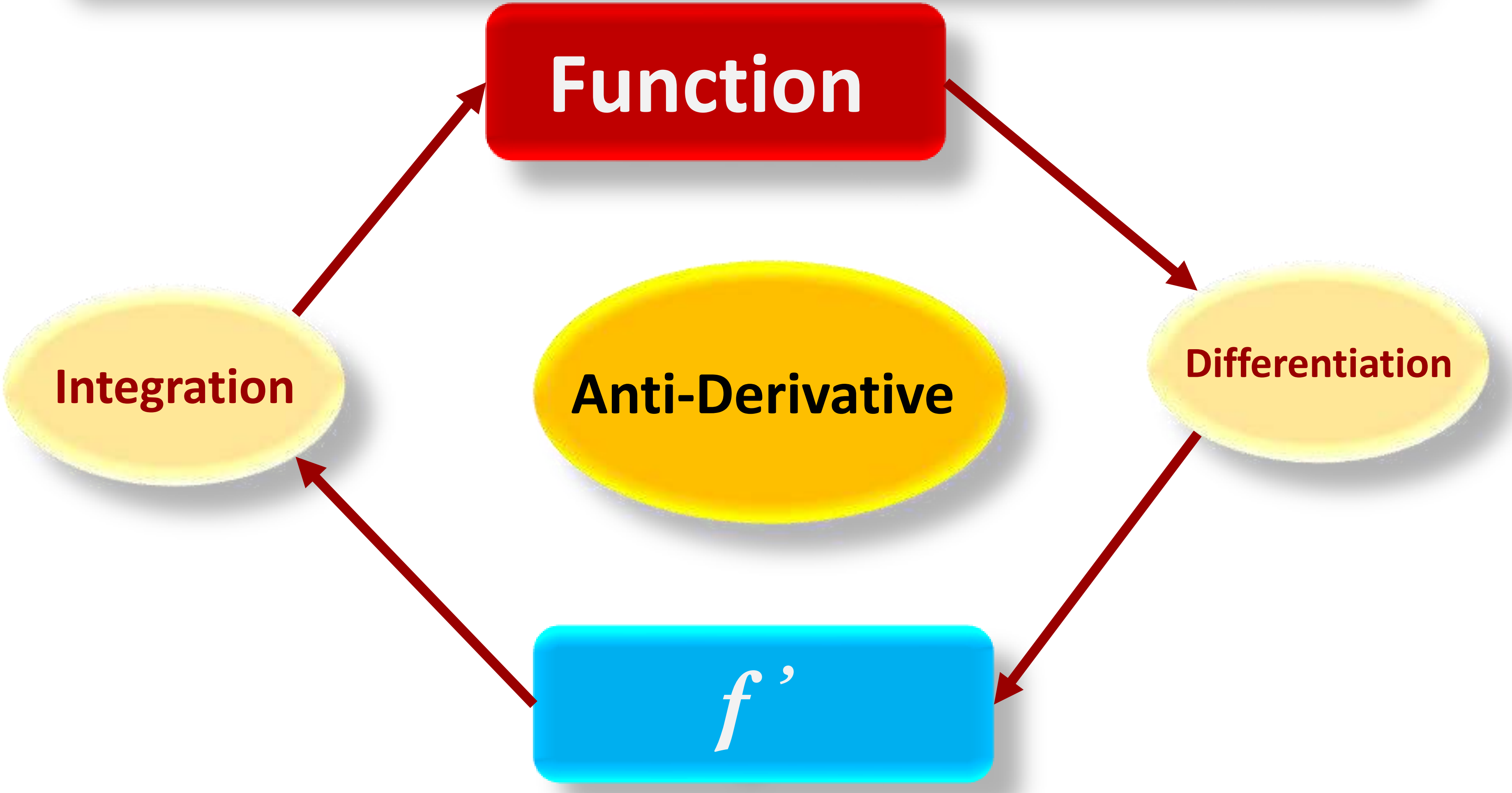
- Meaning, Types-Definite and indefinite integration
- Basic Rule of Integration
- Application of integration in Business and economics- (1) computation of Revenue, Cost function, demand function and profit function from the given function
- (2) Determination of Consumer and Producer Surplus from the function given in both monopoly and pure competition market.
- (3) Learning Curve

INTEGRATION

- Integration is a process of finding a function when its derivative is known… *that's, it is the process reverse to that of differentiation and hence, it is also known as Antiderivative.*

Asha Rani

INTEGRATION...



INTEGRAL CALCULUS...

Integral calculus is the study of the definitions, properties, and applications of two related concepts, ***the indefinite integral* and the *definite integral*.**

The process of finding the value of an integral is called ***Integration*.**

The **Indefinite Integral** is the ***Anti-derivative*,** the inverse operation to the derivative. *F* is an indefinite integral of *f* when *f* is a derivative of *F*.

The **Definite Integral** inputs a function and outputs a number, which gives the area between the graph of the input and the x-axis.

Indefinite Integral…

- It is represented by the following notation:

$$\int f(x)dx$$

- It is viewed as a reverse process of differentiation. Therefore, it is having a value equal to the following with ***unknown C*** because C may take any value. But if some initial conditions are given, then C may assume a specific value.

$$\int f(x)dx = F(x) + C$$

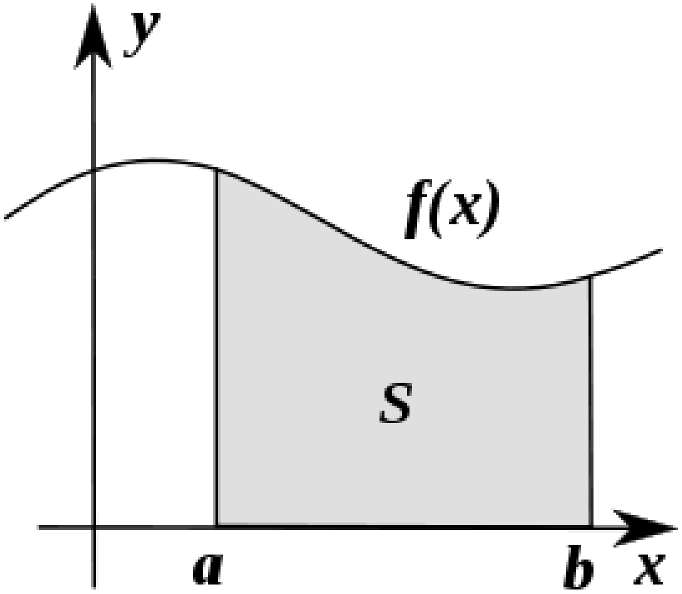
Asha Rani

Definite Integral...

- It is represented by the following notation:

$$\int_a^b f(x)dx$$

- This is nothing but the area under the curve as shown below:



Definite Integral…

- It takes a specific value and free from the variable, x, and an arbitrary constant, C.

- In the following, ***a*** is called the lower limit of the integration while ***b*** is known as the upper limit.

$$\int_a^b f(x)\,dx$$

Asha Rani

Basic Rules of Integration …

Rule No. 1: (The Power Rule)

$$\int x^{n}\,dx = \frac{1}{n+1}x^{n+1} + C$$

Rule No. 2: (The Exponential Rule)

$$\int e^{x}\,dx = e^{x} + C$$

Rule No. 3: (The Logarithmic Rule)

$$\int \frac{1}{x}\,dx = Ln(x) + C$$

Rule No. 4:

$$\int f'(x)\,e^{f(x)}\,dx = e^{f(x)} + C$$

Rule No. 5:

$$\int \frac{f'(x)}{f(x)}\,dx = Ln(f(x)) + C$$

Basic Rules of Operation of Integration ...

Rule No. 6: (The Integral of a Sum)

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx + C$$

Rule No. 7: (The Integral of a Multiple)

$$\int k\, f(x)dx = k \int f(x)dx + C$$

Rule No. 8: (The Substitution Rule)

$$\int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + C$$

Rule No. 9: (The Integration by Parts)

$$\int [f(x)g(x)]dx = f(x)\int g(x)dx + \int f'(x)\int g(x)dx + C$$

Time to apply...

Illustrative Problems :

1. The marginal cost function of firm is MC = $(\log x)^2$. Find the total cost of 100 units if the cost of producing one unit is Rs.22.

2. The additional cost (in lakhs of rupees) of producing a motor car is given by $6 + 4x^2 + 1.5e^{-x}$, where x is the quantity produced. Determine the total cost producing 5 motor cars if the fixed cost is Rs. 7 lakhs. It is given that $e^{-5} = 0.006$.

Solution : 1. $MC = (\log x)^2$

From MC function we have to find cost function

$$\frac{dc}{dx} = (\log x)^2$$

$$dc = (\log x)^2 \, dx$$

by integrating both the sides

$$\int dc = \int (\log x)^2 \cdot 1 \, dx$$

↳ Integrating by parts

$$c = (\log x)^2 \cdot x - \int \left(\frac{2 \log x}{x} \cdot x \right) dx$$

$$= x(\log x)^2 - 2 \int (\log x) \cdot 1 \, dx$$

$$= x(\log x)^2 - 2 \left[\log x \cdot x - \int \frac{1}{x} \cdot x \, dx \right]$$

$$= x(\log x)^2 - 2 \left[x \log x - x \right] + k$$

$$= x(\log x)^2 - 2x \log x - 2x + k$$

k = constant term of integration

cost of producing 1 unit is given i.e ₹22

$$C = 1(\log 1)^2 - 2(1)\log 1 - 2 + k$$

$$k = 20$$

∴ Cost function is

$$C = x(\log x)^2 - 2x \log x + 2x + 20$$

Solution : 2

additional cost is nothing but the Marginal cost

$MC = 6 + 4x^2 + 1.5e^{-x}$

$\frac{dC}{dx} = 6 + 4x^2 + 1.5e^{-x}$

$\int dC = \int (6 + 4x^2 + 1.5e^{-x}) \, dx$

$C = 6x + \frac{4x^3}{3} \mp 1.5e^{-x} + K$

FC = ₹ 7 Lakhs irrespective of no. of units

So, when $x = 0$, $C = 7$ lakhs

$7 = 6 \times 0 + \frac{4(0)^3}{3} - 1.5e^{-0} + K$

$K = 8.5$

$C = 6x + \frac{4x^3}{3} - 1.5e^{-x} + 8.5$

cost of producing 5 motor cars

$C = 6 \times 5 + \frac{4(5)^3}{3} - 1.5e^{-5} + 8.5$

$C = 30 + \frac{500}{3} - 1.5 \times .006 + 8.5$

$C = 205.16$ lakhs of rupees.

Asha Rani

Illustrative Problems…

3. If the Marginal revenue function for output x is given by $MR = \frac{4}{(2x+3)^2} - 1$, find the total revenue function and the demand function.

4. The price elasticity of demand for a commodity is $\pi_d = \frac{3p}{(p-1)(p+2)}$. Find the corresponding demand function if the quantity demanded is 8 units when p = Rs.2.

Sol^n:- 3

$$MR = \frac{4}{(2x+3)^2} - 1$$

$$\frac{dR}{dx} = \frac{4}{(2x+3)^2} - 1$$

$$\int dR = \int\left[\frac{4}{(2x+3)^2} - 1\right] dx$$

$$R = 4 \cdot \frac{(2x+3)^{-2+1}}{-2} - x + K$$

$$R = \frac{-2}{(2x+3)} - x + K$$

when production is 0, Revenue is also 0.
by putting this in R - function

$R = 0 \quad x = 0$

$$0 = \frac{-2}{0+3} - 0 + K \Rightarrow K = \frac{2}{3}$$

$$\therefore R = \frac{-2}{(2x+3)} - x + \frac{2}{3}$$

$$AR = \frac{R}{x} = \frac{-2}{x(2x+3)} - \frac{x}{x} + \frac{2}{3x}$$

$$\boxed{AR = \frac{4}{(6x+9)} - 1}$$

Solution: 4.

$$\lambda d = \frac{3p}{(p-1)(p+2)}$$

$$\lambda d = \frac{-p}{x}\frac{dx}{dp}$$

$$\frac{-p}{x}\frac{dx}{dp} = \frac{3p}{(p-1)(p+2)}$$

$$\int \frac{dx}{x} = \int \frac{-3p}{(p-1)(p+2)p}\,dp$$ → Solving this using partial fraction

$$\frac{-3p}{(p-1)(p+2)p}\,dp = \int \frac{-3}{(p-1)(p+2)}\,dp$$

$$\frac{-3}{(p-1)(p+2)} = \frac{A}{(p-1)} + \frac{B}{(p+2)} \Rightarrow \frac{A(p+2)+B(p-1)}{(p-1)(p+2)}$$

For A, $p-1=0$ $\quad p=1$

$$3A = -3$$
$$A = -1$$

For B, $p+2=0$ $\quad p=-2$

$$-3B = -3$$
$$B = 1$$

$$\int \frac{dx}{x} = \int \left[\frac{-1}{(p-1)} + \frac{1}{(p+2)}\right] dp \Rightarrow \log x = -\log(p-1) + \log(p+2) + \log k$$

$$\log x = \log\frac{(p+2)}{(p-1)} + \log k$$

$$\log x = \log k\frac{(p+2)}{p-1} \Rightarrow x = k\frac{(p+2)}{(p-1)}$$

$x = 8$ when $p = 2$

$k = 2$

$$\boxed{x = \frac{2(p+2)}{p-1}}$$

5.Suppose that when an industrial machine is t years old, it generates revenue at the rate of $R'(t) = 6025 - 8t^2$ rupees per year and results in costs the accumulate at the rate of $C'(t) = 4681 + 13t^2$ rupees per year.

- For how many years is the use of machine profitable?
- What are the net earnings generated by the machine during its period of profitability?

Solⁿ:-5

(a) For finding the no. of years for the profitable use of machine we will equate the

$$R'(t) = C'(t)$$

$$6025 - 8t^2 = 4681 + 13t^2$$

$$1344 - 21t^2 = 0$$

$$t^2 = 64 \Rightarrow t = \pm 8$$

$$t = 8 \text{ years}$$

(b) Net earnings generated by the machine during profitability period

$$P(t) = \int_0^8 [R'(t) - C'(t)]\, dt$$

$$= \int_0^8 (1344 - 21t^2)\, dt$$

$$= \left[1344t - \frac{21t^3}{3}\right]_0^8$$

$$= \left[1344(8) - \frac{21(8)^3}{3} - (1344 \times 0 - 7(0)^3)\right] = \boxed{7168}$$

Consumer Surplus

Consumer surplus is the difference between consumer willingness to pay and the actual amount paid by the consumer for any goods and services.

- How to compute:

(a) Demand function is represented in terms of quantity

- Consumer surplus = $\int_0^{q_e} f(q)dq - p_e * q_e$

(b) Demand function is represented in terms of price

- Consumer surplus = $\int_{p_e}^{p_2} f(p)\, dp$

Producer Surplus

Producer surplus is the difference between what the producer is willing to receive to supply a goods and the actual amount received.

- How to compute:

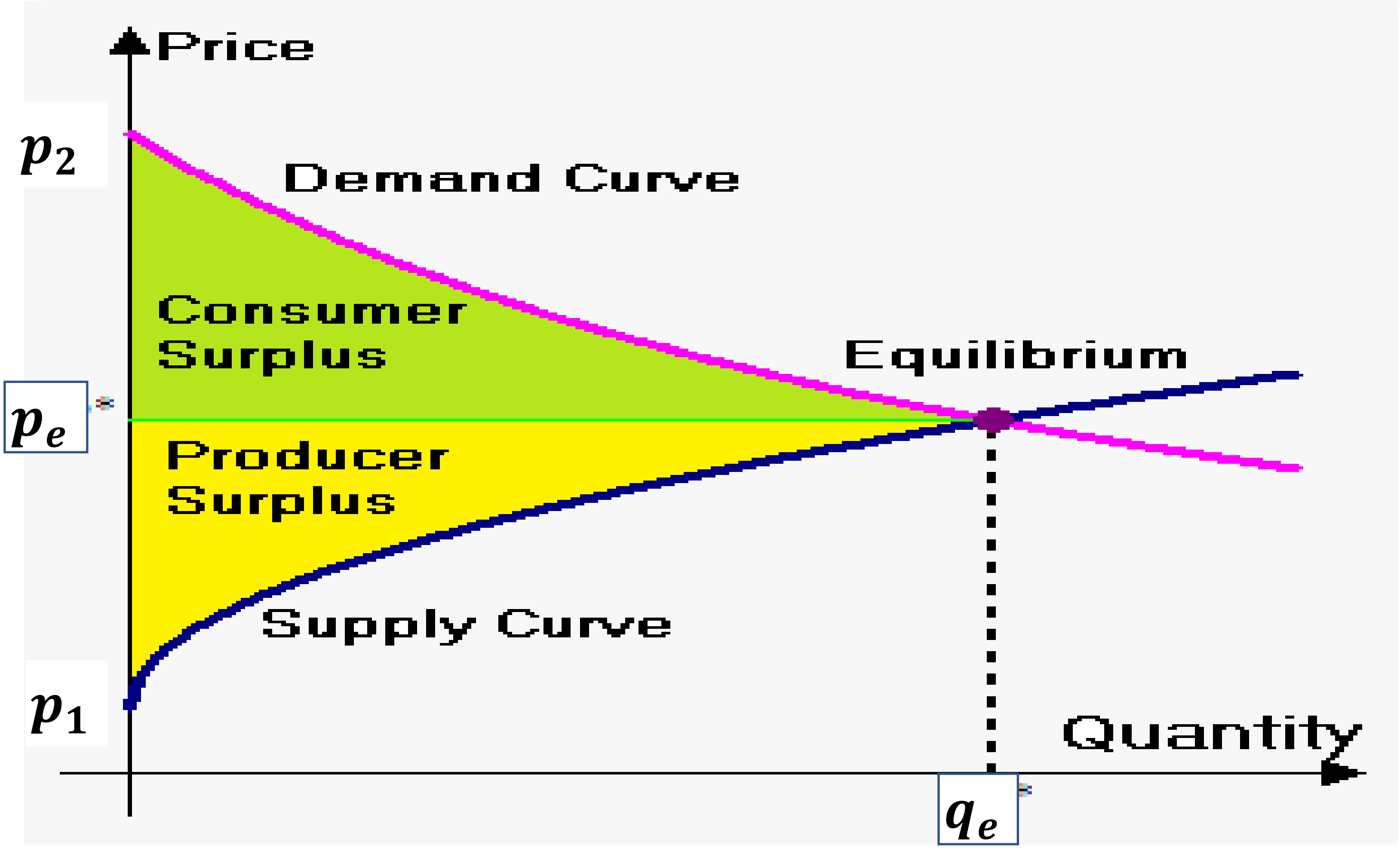
(a) Supply function is represented in terms of quantity

- Producer surplus = $\mathrm{p_e} * \mathrm{q_e} - \int_0^{q_e} g(q)dq$

(b) Supply function is represented in terms of price

- Producer surplus = $\int_{p_1}^{p_e} g(p)\, dp$

Graphical Presentation of Consumer and Producer Surplus



6. Find the consumer and producer' surplus under pure competition, for demand function $p = \frac{8}{x+1} - 2$, and the supply function $p = \frac{1}{2}(x+3)$, where p is the price and x is the quantity.

7. The demand function of a monopolist is $x = 210 - 3p$ and his average cost function is $AC = x + 6 + 10/x$, where p and x refers to the price and quantity of the commodity respectively. Determine consumer surplus at the price and quantity which the monopoly would like to fix.

Solution:- 6.

Under Pure Competition eqm is achieved where

Demand = Supply

$$\frac{8}{(x+1)} - 2 = \frac{1}{2}(x+3)$$

$$x^2 + 8x - 9 = 0$$

$$(x+1)(x+9) = 0$$

$x_0 = 1$ (as -ve value is ignored)

putting $x = 1$ in the demand function to find

$$P_0 = \frac{8}{1+1} - 2 = 2$$

$$C.S = \int_0^{x_0} D(x)\, dx - P_0 \times x_0$$

$$= \int_0^1 \left[\frac{8}{(x+1)} - 2\right] dx - 2\times 1$$

$$= 8\log(x+1) - 2x\Big]_0^1 - 2 \Rightarrow 8\log 2 - 4$$

$$P.S. = P_0 \times x_0 - \int_0^{x_0} S(x)\, dx$$

$$= 2\times 1 - \int_0^1 \frac{1}{2}(x+3)\, dx$$

$$= 2 - \frac{(x+3)^2}{4}\Big]_0^1 = \frac{1}{4}$$

Solution :- 7.

Demand function, $x = 210 - 3p \Rightarrow P = 70 - \frac{x}{3}$

$AC = x + 6 + \frac{10}{x}$

$C = AC \cdot x$

$= x^2 + 6x + 10$

$R = p \cdot x$

= ~~[scribbled out]~~

$= \left(70 - \frac{x}{3}\right) x = 70x - \frac{x^2}{3}$

Under monopoly,

eqm is achieved where MR = MC

$MR = \frac{d}{dx}\left(70x - \frac{x^2}{3}\right) = 70 - \frac{2x}{3}$

$MC = \frac{d}{dx}(x^2 + 6x + 10) = 2x + 6$

MR = MC

$70 - \frac{2x}{3} = 2x + 6$

$x_0 = 24$

$P_0 = 70 - \frac{24}{3} = 62$

and also $\frac{d(MR)}{dx} < \frac{d(MC)}{dx}$

$\therefore$ Π is Maximized at $x_0 = 24$

$$C.S = \int_0^{x_0} D(x)\, dx - P_0 \times x_0$$

$$= \int_0^{24} \left(70 - \frac{x}{3}\right) dx - 62 \times 24 \Rightarrow \left[70x - \frac{x^2}{6}\right]_0^{24} - 1488$$

$= (1680 - 96) - 1488$

$= ₹\, 96$

8. After producing 35 units of a product, the production manager determines that the production facility is following the learning curve of the form $f(x) = 300 - 190e^{-2x}$, where $f(x)$ is the rate of labour hours required to produce xth unit. How many labour hours would be required to produce additional 25 units?

Solution :- 8.

This question belongs to the concept of learning curve.

Learning curve is used to predict future reductions in Labour requirements.

So to find the total number of labour-hours req[d] to produce units numbered 35 through additional 25 units is given by

$$N = \int_{35}^{60} f(x)\, dx$$

$$= \int_{35}^{60} (300 - 190\, e^{-2x})\, dx$$

$$= 300x + 95\, e^{-2x} \Big]_{35}^{60}$$

$$= \left[300(60) + 95(e^{-120})\right] - \left[300(35) - 95e^{-70}\right]$$

= 7500 Labour hours..

Class Discussion Questions of Application of Integration

Ques: 1. The marginal cost function of firm is MC = $(\log x)^2$. Find the total cost of 100 units if the cost of producing one unit is Rs.22.

Ques: 2. A manufacturer's marginal cost function is MC= $0.003x^2 - 0.6x + 40$, where x is the number of units of a product. If x increases from 100 to 200 units, find the total increase in cost.

Ques: 3. The additional cost (in lakhs of rupees) of producing a motor car is given by $6 + 4x^2 + 1.5e^{-x}$, where x is the quantity produced. Determine the total cost producing 5 motor cars if the fixed cost is Rs. 7 lakhs. It is given that $e^{-5} = 0.006$.

Ques: 4. If the Marginal revenue function for output x is given by $MR = \frac{4}{(2x+3)^2} - 1$, find the total revenue function and the demand function.

Ques: 5. A firm's marginal revenue function is MR = $20e^{\frac{-x}{10}}\left(1 - \frac{x}{10}\right)$.Find the corresponding demand function.

Ques: 6. The XYZ co. has approximated the marginal revenue function for one of its products by MR = $20x - 2x^2$ and MC = $81 - 16x + x^2$. Determine the profit maximizing output and the total profit at the optimum level, assuming fixed cost as 0.

Ques: 7. Suppose that when an industrial machine is t years old, it generates revenue at the rate of $R'(t) = 6025 - 8t^2$ rupees per year and results in costs the accumulate at the rate of $C'(t) = 4681 + 13t^2$ rupees per year.

(a) For how many years is the use of machine profitable?
(b) What are the net earnings generated by the machine during its period of profitability?

Ques: 8. Suppose that x years from now, one investment plan will be generating profit at the rate of $P_1(x) = 50 + x^2$ rupees per year, while the second plan will be generating profit at the rate of $P_2(x) = 200 + 5x$ rupees per year. For how many years will the second plan be more profitable one? Compute the net excess profit if invested in the second plan for the period of time in part (i).

Ques: 9. The elasticity of cost is given by $\pi_c = \frac{5x}{2\,(5x+9)}$. Find the total cost function given that the fixed cost is Rs. 18.

Ques: 10. The price elasticity of demand for a commodity is $\pi_d = \frac{3p}{(p-1)(p+2)}$. Find the corresponding demand function if the quantity demanded is 8 units when p = Rs.2.

Ques: 11. The marginal propensity to save is given by $\frac{dS}{dI} = \frac{1}{4} + \frac{1}{6\sqrt{I}}$, where S is savings and I is income. Find the total savings S if S = 12 and I = 25.

Ques: 12. Find the consumer and producer' surplus under pure competition, for demand function $p = \frac{8}{x+1} - 2$, and the supply function $p = \frac{1}{2}(x+3)$, where p is the price and x is the quantity.

Ques: 13. The supply curve for a commodity is $p = \sqrt{9+x}$ and the quantity sold is 7 units. Find the producer's surplus. Can you find the consumer's surplus? If yes, find it, if not explain with the help of diagram why not.

Ques: 14. The price of a washing machine averaged Rs. 4000; ABC Co. Ltd. Sold 20 every month. When the price dropped to an average Rs.1000; 120 were sold every month by the same company. When the price was Rs.4000; 200 machines were available per month for sale. When the price reached Rs.1000, only 50 remained Find the demand and supply functions, assuming that both are linear. Also determine the consumer and the producer's surplus at equilibrium price.

Ques: 15. The demand function of a monopolist is $x = 210 - 3p$ and his average cost function is $AC = x$+ 6 + 10/x, where p and x refers to the price and quantity of the commodity respectively. Determine consumer surplus at the price and quantity which the monopoly would like to fix.

Ques: 16. Assume that in 2016 the annual world use of natural gas was 50 trillion cubic feet. The annual consumption of gas is increasing at a rate of 3 % compounded continuously. How long it takes to use the available gas, if it is known that in 2016 there were 2200 trillion cubic feet proven reserves? Assume that no new discoveries are made. Also find that the total consumption of the first 10 years.

Ques: 17. At t = 0, the annual world use of natural gas was 50 trillion cubic feet. The annual consumption of gas is increasing at the rate of 10% continuously. Find the total consumption for the first 10 years and the consumption of 10^{th} year.

Ques: 18. After an advertisement campaign, the rate of ales of a product is given by S (t) = $1000e^{-0.5t}$, where t is the time in months. Find:

(i) Total cumulative sales after 2 months.

(ii) Sales during the 3rd month; and

(iii) Total sales as a result of the campaign.

Ques: 19. After turning up 50 cars, a company determines that its tuning facility is following a learning curve of the form $f(x) = 1000x^{-1}$, where f(x) is the rate of labour-hours required to tune up the x^{th} unit. How many total labour-hours should they estimate are required to tune up an additional 50 car?

Ques: 20. After producing 35 units of a product, the production manager determines that the production facility is following the learning curve of the form $f(x) = 300 - 190e^{-2x}$, where $f(x)$ is the rate of labour hours required to produce xth unit. How many labour hours would be required to produce additional 25 units?

Ques: 21. The supply function of a producer is given by: $p = 0.4e^{2x}$, where x denotes thousand units. Find the producer's surplus when sale are 2000 units.